# Kinetic Equations

## II. Derivation of Kinetic Equations by Partial Summation

U. BAHR, P. QUAAS, and K. VOSS

Institut für Theoretische Physik der Technischen Universität Dresden

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Ausgehend von einer allgemeinen Bewegungsgleichung für die Einteilchenverteilungsfunktion aus der vorangegangenen Arbeit werden durch Partialsummation und den Grenzübergang $\tau \rightarrow \infty$ kinetische Gleichungen für verdünnte Gase und für Plasmen in Debye-Hückel-Näherung abgeleitet.

### 1. Introduction

The purpose of this paper is to derive kinetic equations for the one-particle distribution function, which are valid for the kinetic state of a many-particle system [1], i. e. equations of the kind

$$i \frac{\partial}{\partial t} n_{1,t} = \boldsymbol{T}\, n_{1,t} \,. \tag{1}$$

A kinetic state means a state in the evolution of a system towards equilibrium, which can completely be described by a time-dependent one-particle distribution function. Therefore the change of the state has to be determined by the state, too, i. e. the operator $\boldsymbol{T}$ has to be time-independent.

The starting point is the general equation of motion (I.5.6) for the one-particle distribution function as derived in the preceding paper [2]

$$i \frac{\partial}{\partial t} n_{1,t} = \boldsymbol{l}_1 n_{1,t} + \int \mathrm{d}2\, \boldsymbol{l}_{12} [n_{1,t} n_{2,t} + \exp\{-i\tau(\boldsymbol{l}_1 + \boldsymbol{l}_2)\} \, \Gamma_{\tau,12}] \,,$$
$$n_{1,t_0} = \exp\{i \tau \boldsymbol{l}\} [n_{1,t} + \Gamma_{-\tau,1}] \tag{2}$$

with the symbols of paper I, where

$$\Gamma_{\tau,12} = \Gamma_{\tau,12}[n_{l,t_0}] \,, \qquad \Gamma_{-\tau} = \Gamma_{-\tau}[n_{l,t}] \tag{3}$$

are functionals of the one-particle distribution function at initial time $t_0$ resp. at time $t$. Equation (2) is different from the desired Eq. (1) because of the explicit time-dependence of the graph-sum $\boldsymbol{\Gamma}_\tau$ which gives a time-dependence of $\boldsymbol{T}$. Moreover the graph-sum is too complicated to be summarized analytically so that introducing one equation of (2) into the other is not possible exactly.

For an approximate evaluation of the Eqs. (2) for special systems in part 2, partial summations based on a dimensional analysis are performed, and in part 3 the limit $\tau \equiv t - t_0 \rightarrow \infty$ will be taken. By this, a general scheme for the derivation of kinetic equations is given, which will be demonstrated by the derivation of the Boltzmann-equation [1] and of the generalized Boltzmann-equation [3] for dilute gases and both the Vlasov-equation and the LENARD-BALESCU-equation [4] for plasmas in part 4.

### 2. Partial Summation

#### 2.1. Selection of Graphs

In the partial summation one selects the graphs essential for a special system, sums them up, and neglects the remaining ones. In order to ascertain the dependence of the selection of graphs on special systems, a dimensional analysis will be used [5]. In a dimensional analysis a factor is separated from every appearing quantity. The factor characterizes the value of the quantity and is represented by convenient system-parameters, so that the rest becomes dimensionless and nearly equal to one. Thereby the values may be estimated in orders of magnitude.

With the parameters of the classical many-body system

$r_0$ = mean range of the interaction,
$p_0$ = $\sqrt{m k T}$ = mean thermal momentum ($m$ = mass of particles, $k$ = Boltzmann-constant, $T$ = mean temperature of the system),
$\mathcal{N}$ = mean density of particles,
$u_0$ = mean strength of the potential of internal interaction

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[2] U. BAHR, P. QUAAS, and K. VOSS, Z. Naturforsch. 23 a, 633 [1968]; preceding paper, hereafter referred to as I.
[3] E. G. D. COHEN, Physica 28, 1025, 1045, 1060 [1960].
[4] A. LENARD, Ann. Phys. N.Y. 10, 390 [1960]. — R. BALESCU, Phys. Fluids 3, 52 [1960]. — R. L. GUERNSEY, Thesis, University of Michigan 1960.
[5] W. POMPE, Ann. Phys. Leipzig 20, 326 [1968].

all expressions may be constructed from the following scheme (dimensionles quantities are marked by a prime):

$$\mathfrak{r} = r_0\,\mathfrak{r}', \quad \mathfrak{p} = p_0\,\mathfrak{p}', \qquad \tau_0 = \frac{m\,r_0}{p_0},$$

$$\int \mathrm{d}s = r_0^3\,p_0^3 \int \mathrm{d}s' \quad \text{resp.} \quad \int \mathrm{d}r = V\,p_0^3 \int \mathrm{d}r',$$

$$\int \mathrm{d}t_s = \tau_0 \int \mathrm{d}t_s' \quad \text{resp.} \quad \int \mathrm{d}t_r = \tau \int \mathrm{d}t_r', \tag{1}$$

$$\frac{\delta}{\delta\eta_s} = \frac{1}{r_0^3\,p_0^3}\,\frac{\delta}{\delta\eta_s'} \quad \text{resp.} \quad \frac{\delta}{\delta\eta_r} = \frac{1}{V\,p_0^3}\,\frac{\delta}{\delta\eta_r'},$$

$$u_{12}(|\mathfrak{r}_1 - \mathfrak{r}_2|) = u_0\,u_{12}'(|\mathfrak{r}_1' - \mathfrak{r}_2'|)\,.$$

The coordinate-integrations yield a factor $r_0^3$, if the integrand contains the interaction which is different from zero only in the range $r_0$, if it doesn't they yield the total volume $V$. By the time-integration a factor $\tau_0$ appears, if the integrand decreases in the interaction-interval $\tau_0$, if it doesn't a factor $\tau$ appears. If the functional derivatives $\delta/\delta\eta_r$ may be replaced by distribution functions, one gets $\mathcal{N}\,p_0^{-3}$. However, applying them to expressions containing the interaction results in $r_0^{-3}\,p_0^{-3}$. Because of the dependence of the interaction on the difference-coordinates, however, only one derivative yields $V^{-1}\,p_0^{-3}$.

According to these considerations one can express the sum of graphs as

$$\boldsymbol{\Gamma}_\tau = N \sum_\alpha^{1\ldots\infty} \sum_\beta^{2\ldots\alpha+1} \sum_\gamma^{0\ldots\alpha} (u_0/k\,T)^\alpha\,(\mathcal{N}\,r_0^3)^{\beta-1}\,(\tau/\tau_0)^\gamma\,\boldsymbol{\Gamma}_\tau'^{(\alpha,\beta,\gamma)} \tag{2}$$

and its first and second derivatives as

$$\boldsymbol{\Gamma}_{\tau,1} \equiv \frac{\delta}{\delta\eta_1}\,\boldsymbol{\Gamma}_\tau = (\mathcal{N}\,p_0^{-3}) \sum_\alpha^{1\ldots\infty} \sum_\beta^{2\ldots\alpha+1} \sum_\gamma^{0\ldots\alpha} (u_0/k\,T)^\alpha (\mathcal{N}\,r_0^3)^{\beta-1}\,(\tau/\tau_0)^\gamma\,\boldsymbol{\Gamma}_{\tau,1}'^{(\alpha,\beta,\gamma)} \tag{3}$$

and

$$\boldsymbol{\Gamma}_{\tau,12} \equiv \frac{\delta^2}{\delta\eta_1\,\delta\eta_2}\,\boldsymbol{\Gamma}_\tau = (\mathcal{N}\,p_0^{-3})^2 \sum_\alpha^{1\ldots\infty} \sum_\beta^{2\ldots\alpha+1} \sum_\gamma^{0\ldots\alpha} (u_0/k\,T)^\alpha\,(\mathcal{N}\,r_0^3)^{\beta-2}\,(\tau/\tau_0)^\gamma\,\boldsymbol{\Gamma}_{\tau,12}'^{(\alpha,\beta,\gamma)}. \tag{4}$$

Each topological type of graphs in $\boldsymbol{\Gamma}_\tau$ ($\alpha$ = number of interactions-lines, $\beta$ = number of ends of arrows) has a distinct power of the coupling constant $u_0/k\,T$ and of the dilution parameter $\mathcal{N}\,r_0^3$ and may have terms with different time-dependence $(\tau/\tau_0)^\gamma$. Since the parameter $\tau/\tau_0$ characterizes the time-behaviour, which is of interest only in the summation of the selected graphs, the factor $(\tau/\tau_0)^\gamma$ is disregarded here. It will be taken into account in a following paper[6] in connection with possible divergencies. The types of graphs are summarized in Table 1, where some characteristic graphs of $\boldsymbol{\Gamma}_\tau$ are also given.

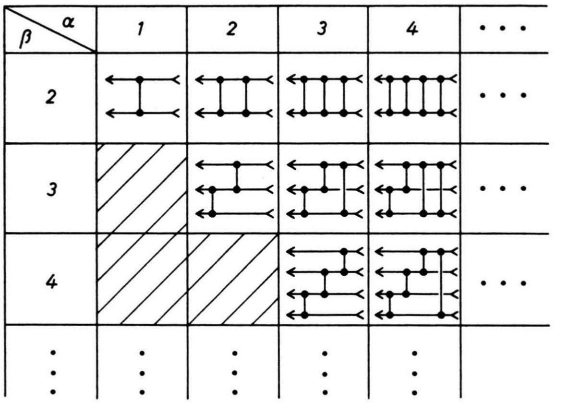


Table 1. Topological types of $\boldsymbol{\Gamma}_\tau$. Below the diagonal there are no graphs.

If certain assumptions are made for the coupling constant and the dilution parameter, only a few essential graphs have to be taken into account, while the remaining ones may be neglected. There are two possibilities. On the one hand, one can successively sum the graphs in the rows of Table 1 and thus gets an expansion relative to powers of $\mathcal{N}\,r_0^3$, which is suitable for dilute gases with $\mathcal{N}\,r_0^3 \ll 1$. On the other hand, one can take into consideration all graphs in the diagonals of Table 1, i. e. all graphs with $(\mathcal{N}\,r_0^3 \cdot u_0/k\,T)^n$ for all $n = 1, 2, \ldots$ with additional powers of $u_0/k\,T$. A comparison with the equilibrium[7] shows, that the summation of the diagonal-graphs coresponds to the introduction of the screened Debye-potential and is therefore suitable for plasmas, where $\mathcal{N}\,r_0^3 \cdot u_0/k\,T \approx 1$ and $u_0/k\,T \ll 1$ is valid.

[6] U. Bahr, P. Quaas, and K. Voss, Z. Naturforsch. **23 a**, 644 [1968], following paper.

[7] J. E. Mayer and E. W. Montroll, J. Chem. Phys. **9**, 2 [1941].

### 2.2. Summation of the Row-Graphs

From the equation of motion (I.3.1)

$$i \frac{\partial}{\partial t} \mathbf{S}_\tau = \tilde{\mathbf{L}}^{\mathbf{v}}{}_\tau \mathbf{S}_\tau , \quad \mathbf{S}_0 = \mathbf{1} \tag{5}$$

for the $\mathbf{S}$-operator (I.3.3.)

$$\mathbf{S}_\tau = N_{\eta,\, \delta/\delta\eta} \exp\{\mathbf{\Gamma}_\tau\} , \quad \mathbf{\Gamma}_0 = 0 \tag{6}$$

there follows a differential equation for the linked-cluster sum

$$i \frac{\partial}{\partial t} \mathbf{\Gamma}_\tau = \int \frac{\mathrm{d}1\, \mathrm{d}2}{2!} \eta_1 \eta_2 \tilde{\mathbf{l}}_{12,\tau} N_{\eta,\, \delta/\delta\eta} \left( \frac{\delta}{\delta\eta_1} \frac{\delta}{\delta\eta_2} + \mathbf{\Gamma}_{\tau,1} \frac{\delta}{\delta\eta_2} + \mathbf{\Gamma}_{\tau,2} \frac{\delta}{\delta\eta_1} + \mathbf{\Gamma}_{\tau,1}\mathbf{\Gamma}_{\tau,2} + \mathbf{\Gamma}_{\tau,12} \right). \tag{7}$$

By expanding the linked-cluster sum in powers of the parameters $\mathcal{N} r_0^3$ and $u_0/k T$ and by putting this expansion into (7) one obtains by a comparison of coefficients differential equations for the linked-cluster sum in the desired approximation.

The expansion in powers of $\mathcal{N} r_0^3$ for dilute gases leads to the sum

$$\mathbf{\Gamma}_\tau = \sum_s^{2\ldots\infty} \int \frac{\mathrm{d}1 \ldots \mathrm{d}s}{s!} \eta_1 \ldots \eta_s \mathbf{k}_{1\ldots s,\tau} \frac{\delta}{\delta\eta_1} \cdots \frac{\delta}{\delta\eta_s} , \quad \mathbf{k}_{1\ldots s,0} = 0 , \tag{8}$$

because the number of ends of arrows and the powers of $\mathcal{N} r_0^3$ are connected unequivocally. The coefficient-operators $\mathbf{k}_{1\ldots s,\tau}$ are determined by putting (8) into (7). Differential equations are obtained, as for instance for $\mathbf{k}_{12,\tau}$

$$i \frac{\partial}{\partial t} \mathbf{k}_{12,\tau} = \tilde{\mathbf{l}}_{12,\tau} (1 + \mathbf{k}_{12,\tau}) , \quad \mathbf{k}_{12,0} = 0 \tag{9}$$

with the solution

$$\mathbf{k}_{12,\tau} = \mathbf{\Psi}_{12,\tau} - 1 ,$$
$$\mathbf{\Psi}_{1\ldots s,\tau} \equiv \exp\{i\tau(\mathbf{l}_1 + \ldots + \mathbf{l}_s)\} \exp\{-i\tau \mathbf{l}_{1\ldots s}\} . \tag{10}$$

In the same way, equations for $\mathbf{k}_{123,\tau}$ and $\mathbf{k}_{1234,\tau}$ may be gained with the solutions

$$\mathbf{k}_{123,\tau} = \mathbf{\Psi}_{123,\tau} - \mathbf{k}_{12,\tau} - \mathbf{k}_{13,\tau} - \mathbf{k}_{23,\tau} - 1 \tag{11}$$

and

$$\begin{aligned} \mathbf{k}_{1234,\tau} = {} & \mathbf{\Psi}_{1234,\tau} - \mathbf{k}_{123,\tau} - \mathbf{k}_{134,\tau} - \mathbf{k}_{234,\tau} - \mathbf{k}_{124,\tau} \\ & - \mathbf{k}_{12,\tau}\mathbf{k}_{34,\tau} - \mathbf{k}_{13,\tau}\mathbf{k}_{24,\tau} - \mathbf{k}_{14,\tau}\mathbf{k}_{23,\tau} \\ & - \mathbf{k}_{12,\tau} - \mathbf{k}_{13,\tau} - \mathbf{k}_{14,\tau} - \mathbf{k}_{24,\tau} - \mathbf{k}_{23,\tau} - \mathbf{k}_{34,\tau} - \mathbf{1} . \end{aligned} \tag{12}$$

The general appearance of $\mathbf{k}_{1\ldots s,\tau}$ may be given, but is of no practical interest.

### 2.3. Summation of Diagonal-Graphs

The only important summation of diagonals is the so-called chain-approximation, which takes into account only graphs of $\mathbf{\Gamma}_\tau$ with $(\mathcal{N} r_0^3 \cdot u_0/k T)^n$

$$\mathbf{\Gamma}_\tau \approx \mathbf{\Gamma}_\tau^{(k)} \equiv \sum_n^{1\ldots\infty} \left(\mathcal{N} r_0^3 \frac{u_0}{k T}\right)^n N \sum_\gamma^{0\ldots n} \left(\frac{\tau}{\tau_0}\right)^\gamma \mathbf{\Gamma}_\tau'^{(n,n+1,\gamma)} \tag{13}$$

with those graphs being neglected which yield powers of $u_0/k T$ higher than the powers of $\mathcal{N} r_0^3$. In this approximation a dimensional analysis of (7) shows that the term containing $\mathbf{\Gamma}_{\tau,12}$ may be neglected. After time-integration of (7) an integral-equation

$$\mathbf{\Gamma}_\tau^{(k)} = \int_0^\tau \mathrm{d}t_1 \int \frac{\mathrm{d}1\, \mathrm{d}2}{2!} \eta_1 \eta_2 \tilde{\mathbf{l}}_{12,t_1} N_{\eta,\, \delta/\delta\eta} \left( \frac{\delta}{\delta\eta_1} \frac{\delta}{\delta\eta_2} + \mathbf{\Gamma}_{t_1,1}^{(k)} \frac{\delta}{\delta\eta_2} + \mathbf{\Gamma}_{t_1,2}^{(k)} \frac{\delta}{\delta\eta_1} + \mathbf{\Gamma}_{t_1,1}^{(k)} \mathbf{\Gamma}_{t_1,2}^{(k)} \right) \tag{14}$$

for the linked-cluster sum in chain-approximation has been found.

For the derivation of kinetic equations from (1.2) we also need the integral-equation for the first functional derivative of $\mathbf{\Gamma}_\tau^{(k)}$

$$\begin{aligned} \mathbf{\Gamma}_{\tau,1}^{(k)} = {} & \int_0^\tau \mathrm{d}t_1 \int \mathrm{d}3\, \eta_3 \tilde{\mathbf{l}}_{13,t_1} N_{\eta,\, \delta/\delta\eta} \left( \frac{\delta}{\delta\eta_1} \frac{\delta}{\delta\eta_3} + \mathbf{\Gamma}_{t_1,1}^{(k)} \frac{\delta}{\delta\eta_3} + \mathbf{\Gamma}_{t_1,3}^{(k)} \frac{\delta}{\delta\eta_1} + \mathbf{\Gamma}_{t_1,1}^{(k)} \mathbf{\Gamma}_{t_1,3}^{(k)} \right) \\ & + \int_0^\tau \mathrm{d}t_1 \int \frac{\mathrm{d}3\, \mathrm{d}4}{2!} \eta_3 \eta_4 \tilde{\mathbf{l}}_{34,t_1} N_{\eta,\, \delta/\delta\eta} \left( \mathbf{\Gamma}_{t_1,13}^{(k)} \frac{\delta}{\delta\eta_4} + \mathbf{\Gamma}_{t_1,14}^{(k)} \frac{\delta}{\delta\eta_3} + \mathbf{\Gamma}_{t_1,13}^{(k)} \mathbf{\Gamma}_{t_1,4}^{(k)} + \mathbf{\Gamma}_{t_1,14}^{(k)} \mathbf{\Gamma}_{t_1,3}^{(k)} \right) \end{aligned} \tag{15}$$

and for the second functional derivative

$$\begin{aligned}
\boldsymbol{\Gamma}^{(k)}_{\tau,12} = \int_0^\tau \mathrm{d}t_1\, \tilde{\boldsymbol{l}}_{12,t_1} N_{\eta,\,\delta/\delta\eta} \left(\frac{\delta}{\delta\eta_1}\frac{\delta}{\delta\eta_2} + \boldsymbol{\Gamma}^{(k)}_{t_1,1}\frac{\delta}{\delta\eta_2} + \boldsymbol{\Gamma}^{(k)}_{t_1,2}\frac{\delta}{\delta\eta_1} + \boldsymbol{\Gamma}^{(k)}_{t_1,1}\boldsymbol{\Gamma}^{(k)}_{t_1,2}\right) \\
+ \int_0^\tau \mathrm{d}t_1 \int \mathrm{d}3\,\eta_3\, \tilde{\boldsymbol{l}}_{13,t_1} N_{\eta,\,\delta/\delta\eta} \left(\boldsymbol{\Gamma}^{(k)}_{t_1,12}\frac{\delta}{\delta\eta_3} + \boldsymbol{\Gamma}^{(k)}_{t_1,23}\frac{\delta}{\delta\eta_1} + \boldsymbol{\Gamma}^{(k)}_{t_1,12}\boldsymbol{\Gamma}^{(k)}_{t_1,3} + \boldsymbol{\Gamma}^{(k)}_{t_1,1}\boldsymbol{\Gamma}^{(k)}_{t_1,23}\right) \\
+ \int_0^\tau \mathrm{d}t_1 \int \mathrm{d}3\,\eta_3\, \tilde{\boldsymbol{l}}_{23,t_1} N_{\eta,\,\delta/\delta\eta} \left(\boldsymbol{\Gamma}^{(k)}_{t_1,13}\frac{\delta}{\delta\eta_2} + \boldsymbol{\Gamma}^{(k)}_{t_1,12}\frac{\delta}{\delta\eta_3} + \boldsymbol{\Gamma}^{(k)}_{t_1,13}\boldsymbol{\Gamma}^{(k)}_{t_1,2} + \boldsymbol{\Gamma}^{(k)}_{t_1,12}\boldsymbol{\Gamma}^{(k)}_{t_1,3}\right) \\
+ \int_0^\tau \mathrm{d}t_1 \int \frac{\mathrm{d}3\,\mathrm{d}4}{2!}\,\eta_3\,\eta_4\, \tilde{\boldsymbol{l}}_{34,t_1} N_{\eta,\,\delta/\delta\eta} \left(\boldsymbol{\Gamma}^{(k)}_{t_1,123}\frac{\delta}{\delta\eta_4} + \boldsymbol{\Gamma}^{(k)}_{t_1,124}\frac{\delta}{\delta\eta_3} + 2\,\boldsymbol{\Gamma}^{(k)}_{t_1,123}\boldsymbol{\Gamma}^{(k)}_{t_1,4} + 2\,\boldsymbol{\Gamma}^{(k)}_{t_1,13}\boldsymbol{\Gamma}^{(k)}_{t_1,24}\right).
\end{aligned} \qquad (16)$$

## 3. Limit $\tau \to \infty$

The Eqs. (1.2) are valid for all times, where the assumption of molecular chaos is correct, also in case that an approximated sum of graphs is put into (1.2). For the considerations of part 2 concern neglects for special systems independent of the time-behaviour. But for the kinetic Eq. (1.1) the operator $\boldsymbol{T}$ applied to the one-particle distribution function has to be time-independent. This can be obtained by the limit $\tau \to \infty$, i. e. one assumes the initial time $t_0 \to -\infty$ to be very remote. When no unequivocal limits exist, they are defined with the aid of a convergence factor $\exp\{-\varepsilon\, t_i\}$ in every time-integral. Afterwards, the limit $\varepsilon \to 0$ is taken. This convergence factor practically causes an averaging of the time-behaviour. Thus the observed behaviour in the kinetic state is reproduced with approximate correctness and possible fluctuations are oppressed. In the case of this limit, so-called secular terms may appear which increase with $\lim_{\tau\to\infty} \tau^n$. As to this phenomenon it will be pointed out in the following paper [6] that with partial summation the terms have to be summarized in such a way that the total expression does not diverge.

After summarizing the results of I and the present paper, the desired kinetic equations of the kind (1.1) have been obtained from the Liouville-equation in the following steps:

- reduction of the $N$-particle distribution function,
- molecular chaos at initial time $t_0$,
- calculation of the one-particle distribution function at initial time by the hermitian adjoint time-evolution operator,
- molecular chaos at time $t$ relative to the foregoing procedure,
- selection of graphs for special systems and summation,
- limit $\tau \to \infty$ with a convergence factor.

As these equations are valid only in the kinetic state, the initial conditions for their solution must be contained in the kinetic state.

## 4. Special Kinetic Equations

### *4.1. Boltzmann-Equation*

In the lowest approximation of the expansion of the linked-clusters in $\mathcal{N}\, r_0^3$ (first row of Table 1) one has to sum all ladder graphs. One puts (2.10) with (2.8) into the first equation of (1.2). From the second equation (1.2) the second term with $\boldsymbol{\Gamma}_{-\tau,1}$ need not be considered in this approximation. After taking the limit of part 3, one gets a kinetic equation for dilute gases with $\mathcal{N}\, r_0^3 \ll 1$, the Boltzmann-equation

$$i\frac{\partial}{\partial t} n_{1,t} = \boldsymbol{l}_1\, n_{1,t} + \int \mathrm{d}2\, \boldsymbol{l}_{12}(1+\boldsymbol{\varphi}_{12})\, n_{1,t}\, n_{2,t} \qquad (1)$$

where $\boldsymbol{\varphi}_{12}$ is the following operator

$$\begin{aligned}
\boldsymbol{\varphi}_{12} &\equiv \lim_{\tau\to\infty} \exp\{-i\tau(\boldsymbol{l}_1+\boldsymbol{l}_2)\}\, \boldsymbol{k}_{12,\tau} \exp\{i\tau(\boldsymbol{l}_1+\boldsymbol{l}_2)\} \\
&= \lim_{\tau\to\infty} (\exp\{-i\tau(\boldsymbol{l}_1+\boldsymbol{l}_2+\boldsymbol{l}_{12})\}\exp\{i\tau(\boldsymbol{l}_1+\boldsymbol{l}_2)\} - 1) \\
&= \lim_{\varepsilon\to 0} (-i) \int_0^\infty \mathrm{d}t_1\, e^{-\varepsilon t_1} \exp\{-i\,t_1(\boldsymbol{l}_1+\boldsymbol{l}_2+\boldsymbol{l}_{12})\}\, \boldsymbol{l}_{12} \exp\{i\,t_1(\boldsymbol{l}_1+\boldsymbol{l}_2)\}\,. 
\end{aligned} \qquad (2)$$

In this form (1) the Boltzmann-equation was also derived by Cohen [3].

### 4.2. Generalized Boltzmann-Equation

In the next approximation relative to powers of $\mathcal{N} r_0^3$ the graphs of the second row in Table 1 are incorporated in the sum of graphs. Putting the second equation of (1.2) into the first the term which is linear in $\mathcal{N} r_0^3$ from $\boldsymbol{\Gamma}_{-\tau,1}$ has to be accounted for. One gets a generalized Boltzmann-equation for moderately dense gases

$$i \frac{\partial}{\partial t} n_{1,t} = \boldsymbol{l}_1 n_{1,t} + \int d2\, \boldsymbol{l}_{12} [(\boldsymbol{1} + \boldsymbol{\varphi}_{12}) + \int d3\, (\boldsymbol{\varphi}_{123} + \boldsymbol{\varphi}_{12} \boldsymbol{\varphi}_{13}^{\dagger} + \boldsymbol{\varphi}_{12} \boldsymbol{\varphi}_{23}^{\dagger} - \boldsymbol{\varphi}_{12} - \boldsymbol{\varphi}_{23} - \boldsymbol{\varphi}_{13})\, n_{3,t}]\, n_{1,t}\, n_{2,t}. \quad (3)$$

Corresponding to (2) the triple-collision operator $\boldsymbol{\varphi}_{123}$ means

$$\boldsymbol{\varphi}_{123} \equiv \lim_{\tau \to \infty} (\exp\{-i\tau \boldsymbol{l}_{123}\} \exp\{i\tau(\boldsymbol{l}_1 + \boldsymbol{l}_2 + \boldsymbol{l}_3)\} - \boldsymbol{1})$$

$$= \lim_{\varepsilon \to 0} (-i) \int_0^\infty dt_1\, e^{-\varepsilon t_1} \exp\{i t_1 \boldsymbol{l}_{123}\} (\boldsymbol{l}_{12} + \boldsymbol{l}_{23} + \boldsymbol{l}_{13}) \exp\{i t_1 (\boldsymbol{l}_1 + \boldsymbol{l}_2 + \boldsymbol{l}_3)\}. \quad (4)$$

Moreover (3) contains the hermitian adjoint operator $\boldsymbol{\varphi}_{12}^{\dagger}$

$$\boldsymbol{\varphi}_{12}^{\dagger} \equiv \lim_{\tau \to \infty} (\exp\{-i\tau(\boldsymbol{l}_1 + \boldsymbol{l}_2)\} \exp\{i\tau(\boldsymbol{l}_1 + \boldsymbol{l}_2 + \boldsymbol{l}_{12})\} - \boldsymbol{1})$$

$$= \lim_{\varepsilon \to 0} i \int_0^\infty dt_1\, e^{-\varepsilon t_1} \exp\{-i t_1 (\boldsymbol{l}_1 + \boldsymbol{l}_2)\}\, \boldsymbol{l}_{12} \exp\{i t_1 (\boldsymbol{l}_1 + \boldsymbol{l}_2 + \boldsymbol{l}_{12})\}. \quad (5)$$

The form (3) of the generalized Boltzmann-equation derived here differs from the common one [3] because of the different method used to eliminate the initial state $n_{1,t_0}$ by $n_{1,t}$. So far nothing can be said about the physical meaning of this difference.

### 4.3. Vlasov-Equation

For dilute plasmas with $\mathcal{N} r_0^3 \cdot u_0/kT \approx 1$ and $u_0/kT \ll 1$ one takes all graphs in $\boldsymbol{\Gamma}_{\tau,1}$ with factors $(\mathcal{N} r_0^3 \cdot u_0/kT)^n$ for all $n = 1, 2, \ldots$ Using in

$$n_{1,t} = \exp\{-i\tau \boldsymbol{l}_1\} [n_{1,t_0} + \Gamma_{\tau,1}] \quad (6)$$

for $\Gamma_{\tau,1}$

$$\Gamma_{\tau,1} \equiv P\, \boldsymbol{\Gamma}_{\tau,1}^{(k)} \left[ \frac{\delta}{\delta \eta_s} = n_{s,t_0} \right] \quad (7)$$

and the integral equation (2.15) one gets

$$n_{1,t} = \exp\{-i\tau \boldsymbol{l}_1\} [n_{1,t_0} + \int_0^\tau dt_1 \int d2\, \tilde{\boldsymbol{l}}_{12,t_1} n_{1,t_0} n_{2,t_0} + \Gamma_{t_1,1} n_{2,t_0} + \Gamma_{t_1,2} n_{1,t_0} + \Gamma_{t_1,1} \Gamma_{t_1,2}]. \quad (8)$$

This leads to an iteration-equation for the one-particle distribution function

$$n_{1,t} = \exp\{-i\tau \boldsymbol{l}_1\} [n_{1,t_0} + \int_0^\tau dt_1 \int d2\, \tilde{\boldsymbol{l}}_{12,t_1} \times \exp\{i t_1 (\boldsymbol{l}_1 + \boldsymbol{l}_2)\}\, n_{1,t_1} n_{2,t_1}]. \quad (9)$$

After time-differentiation and with the definition of (I.3.1) the Vlasov-equation for dilute plasma arises from (8)

$$i \frac{\partial}{\partial t} n_{1,t} = \boldsymbol{l}_1 n_{1,t} + \int d2\, \boldsymbol{l}_{12} n_{1,t} n_{2,t}. \quad (10)$$

In this case, the limit $\tau \to \infty$ need not be taken, because the operator $\boldsymbol{T}$ is already time-independent.

### 4.4. Lenard-Balescu-Equation

A corrected kinetic equation for plasmas also accounting for collective effects (screening) is obtained by an additional summing in the graph-sum $\boldsymbol{\Gamma}_\tau$ or $\boldsymbol{\Gamma}_{\tau,1}$ of all graphs with factors $(u_0/kT)(\mathcal{N} r_0^3 \cdot u_0/kT)^n$ for all $n = 1, 2, \ldots$, i.e. the graphs of the next diagonal in Table 1. Because of the dimensional relation (2.4), this corresponds to a summation of all graph in chain-approximation for $\boldsymbol{\Gamma}_{\tau,12}$. This is analogous to the equilibrium [7] where the simple chain-approximation for the radial distribution function corresponds to the Debye-Hückel-approximation for the partition function. We start from the iteration equation (2.16). According to Balescu [5] one has to make the ansatz

$$n_{1,t} n_{2,t} \ldots n_{s,t} \approx n_{1,t} n_{2',t} \ldots n_{s',t}, \quad s' \equiv \mathfrak{r}_1 \mathfrak{p}_s \quad (11)$$

for the corrective term. By summing up all terms with $(\mathcal{N} r_0^3 \cdot u_0/kT)^n$ the screening of the potential at the Debye-length has been taken into account. When this length is large (dilute plasma) the plasma is known to be well described by the Vlasov term, i. e. the corrective term is then approximately equal to zero. For small Debye lengths, i. e. when the particles have to come very close to one another in order to be able to interact, the corrective term becomes effective. For this reason the above estimation (11) is admissible.

With the ansatz (11) in (1.2) and (2.16) because of

$$\int \mathrm{d}2\, \boldsymbol{l}_{12}\, n_{1,t}\, n_{2',t} = 0 \qquad (12)$$

(after partial integration relative to $\mathfrak{r}_2$) many terms can be omitted, particularly

$$\Gamma^{(k)}_{-\tau,1} \approx 0 \qquad (13)$$

is valid. By this, the more simple integral-equation

$$\Gamma^{(k)}_{\tau,12} = \int_0^\tau \mathrm{d}t_1\, \tilde{\boldsymbol{l}}_{12,t_1}\, n_{1,t_0}\, n_{2',t_0} + \int_0^\tau \mathrm{d}t_1 \int \mathrm{d}3\, [\tilde{\boldsymbol{l}}_{13,t_1}\, \Gamma^{(k)}_{t_1,23}\, n_{1,t_0} + \tilde{\boldsymbol{l}}_{23,t_1}\, \Gamma^{(k)}_{t_1,13} n_{2',t_0}]\,,$$

$$n_{s',t_0} = \exp\{i\,\tau\, \boldsymbol{l}_s\}\, n_{s',t} \qquad (14)$$

follows from (2.16). Performing the limit $\tau \to \infty$ as in part 3 and defining

$$\Gamma^{(k)}_{t,12} \equiv \exp\{i\,t(\boldsymbol{l}_1 + \boldsymbol{l}_2)\}\, \boldsymbol{a}_{12} \times \exp\{i\,\tau(\boldsymbol{l}_1 + \boldsymbol{l}_2)\}\, n_{1,t}\, n_{2',t} \qquad (15)$$

one obtains from (1.2) a kinetic equation, the Lenard-Balescu equation, for the plasma in Debye-Hückel approximation

$$i\frac{\partial}{\partial t} n_{1,t} = \boldsymbol{l}_1\, n_{1,t} + \int \mathrm{d}2\, \boldsymbol{l}_{12}\, n_{1,t}\, n_{2,t} + \int \mathrm{d}2\, \boldsymbol{l}_{12}\, \boldsymbol{a}_{12} n_{1,t}\, n_{2',t}\,. \qquad (16)$$

For the operator $\boldsymbol{a}_{12}$ the integral-equation

$$\boldsymbol{a}_{12} = \boldsymbol{\delta}^{12}_{-}\, \boldsymbol{l}_{12} \,| + \boldsymbol{\delta}^{12}_{-} \int \mathrm{d}3\, (\boldsymbol{l}_{13}\, \boldsymbol{a}_{23} + \boldsymbol{l}_{23}\, \boldsymbol{a}_{13}) \,|\, n_{3',t} \qquad (17)$$

arises with (15) after the limit from (14). In this, the abbreviation

$$\boldsymbol{\delta}^{12}_{-} \equiv \lim_{\varepsilon \to 0}\, (-i) \int_0^\infty \mathrm{d}t_1 \exp\{-i\,t_1(\boldsymbol{l}_1 + \boldsymbol{l}_2) - \varepsilon\, t_1\} \qquad (18)$$

was used. In (17) the operators $\boldsymbol{l}_i$ contained in (18) must only be applied up to the vertical line. With the definitions

$$u_{12}(|\mathfrak{r}_1 - \mathfrak{r}_2|) \equiv \int \frac{\mathrm{d}\mathfrak{k}}{(2\,\pi)^{3/2}} \exp\{i\,\mathfrak{k}(\mathfrak{r}_1 - \mathfrak{r}_2)\}\, u_k;$$

$$u_k = u_k^*; \quad k \equiv |\mathfrak{k}|;$$

$$\boldsymbol{l}_{12} = -\int \frac{\mathrm{d}\mathfrak{k}}{(2\,\pi)^{3/2}} \exp\{i\,\mathfrak{k}(\mathfrak{r}_1 - \mathfrak{r}_2)\}\, u_k\, \mathfrak{k}\, \vec{\partial}_{12};$$

$$\vec{\partial}_{12} \equiv \frac{\partial}{\partial \mathfrak{p}_1} - \frac{\partial}{\partial \mathfrak{p}_2}\,; \qquad \vec{\partial}_1 \equiv \frac{\partial}{\partial \mathfrak{p}_1}\,; \qquad (19)$$

$$\delta^{13,\mathfrak{k}}_{-} \equiv \lim_{\varepsilon \to 0}\, (-i) \int_0^\infty \mathrm{d}t' \exp\left(-i\,t'\, \frac{\mathfrak{p}_1 - \mathfrak{p}_3}{m}\, \mathfrak{k} - \varepsilon\, t'\right)$$

the same integral equation as given and solved by Balescu [5] is obtained by Fourier transformation (17). The closed form of the Lenard-Balescu equation runs as follows:

$$i\frac{\partial}{\partial t} n_{1,t} = \boldsymbol{l}_1\, n_{1,t} + \int \mathrm{d}2\, \boldsymbol{l}_{12}\, n_{1,t}\, n_{2,t} + i \int \mathrm{d}\mathfrak{k}\, \mathrm{d}\mathfrak{p}_2\, \mathfrak{k}\, \vec{\partial}_1 \left|\frac{u_k}{\varepsilon_{1,\mathfrak{k}}}\right|^2 \times \delta\left(\frac{\mathfrak{p}_1 - \mathfrak{p}_2}{m}\, \mathfrak{k}\right) \mathfrak{k}\, \vec{\partial}_{12}\, n_{1,t}\, n_{2',t}\,, \qquad (20)$$

$$\varepsilon_{1,\mathfrak{k}} \equiv 1 - (2\,\pi)^{3/2} \int \mathrm{d}\mathfrak{p}_3\, \delta^{13,\mathfrak{k}}_{-}\, u_k\, \mathfrak{k}\, \partial_3\, n_{3',t}\,.$$

The factor $\varepsilon_{1,\mathfrak{k}}$ expresses a non-stationary screening.

Gould and de Witt [8] remarked that with the numerical evaluation of the expressions in the Lenard-Balescu equation correct results for transport coefficients failed to be obtained. Therefore they performed a further partial summation allowing more adequately for the actual interaction at short distances between particles in a plasma.

[8] H. A. Gould and H. E. de Witt, Phys. Rev. 155, 68 [1967].